Projective Characters of Finite Chevalley Groups

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1. INTRODUCTION

1.1 Notation

We begin by reviewing standard results and establishing notation. (See [10] for general reference.)

Let p be a prime number and let m denote a positive integer or ∞. For $m<\infty$, set $q=p^m$ and let $\mathbb{F}_q$ be a field of order q; for $m=\infty$, set $q=\infty$ and let $\mathbb{F}_\infty=K$ be an algebraic closure of $\mathbb{F}_p$.

Fix an irreducible root system Ψ of rank ℓ and let $G^{(m)}$ denote the universal Chevalley group of type Ψ defined over $\mathbb{F}_q$. $G^{(\infty)}$ is an algebraic group and for $m<\infty$, $G^{(m)}$ is a finite subgroup of $G^{(\infty)}$.

Choose a system $\{\alpha_i, 1\leq i\leq\ell\}$ of simple roots in Ψ and let $\{\lambda_i, 1\leq i\leq\ell\}$ be the corresponding fundamental dominant weights. (For definiteness we assume that the α_i have been numbered according to the usual labeling of the vertices in the associated Dynkin diagram (see Humphreys [7], p. 58).) The λ_i form a $\mathbb{Z}$-basis for the weight lattice Λ associated with Ψ. A partial order $\prec$ is defined on Λ by setting $\lambda\prec\mu$ if $\mu-\lambda \in \sum \mathbb{Z}^+\alpha_i$. For $n\in\mathbb{Z}^+$ set $\Lambda_n = \{\sum a_i\lambda_i \in$

$\Lambda | 0 \leq a_i < n\}$ and let $\Lambda_\infty = \Lambda^+$ denote the set $\sum \mathbb{Z}^+\lambda_i$ of dominant weights.

From this point on, we fix $1 \leq m \leq \infty$ and set $G = G^{(m)}$. By "G-module" we shall mean finite dimensional KG-module if $m < \infty$ and finite dimensional rational G-module if $m = \infty$.

The irreducible G-modules are indexed by Λ_q as follows: The irreducible $G^{(\infty)}$-modules are indexed by Λ^+ via highest weights and those modules with indices in Λ_q remain irreducible upon restriction to G and form a complete set of pairwise nonisomorphic irreducible G-modules. Let $M(\lambda)$ denote the irreducible G-module associated with $\lambda \in \Lambda_q$ in this fashion.

Given any G-module M we denote by $Fr(M)$ the G-module which has the same underlying vector space as M but on which $g \in G$ acts according to the new rule $g \cdot x = Fr(g)x$ $(x \in M)$ where Fr is the Frobenius automorphism of G which raises matrix entries to the pth power. It is easy to see that for $\lambda \in \Lambda_p$ and $0 \leq j < m-1$ we have $Fr(M(p^j\lambda)) \simeq M(p^{j+1}\lambda)$ while $Fr(M(p^{m-1}\lambda)) \simeq M(\lambda)$ if $m < \infty$.

1.2 Purpose and Method

We now state the main tool used in the paper.

1.2.1 STEINBERG'S TENSOR PRODUCT THEOREM ([10], p. 217). *Let* $\mu \in \Lambda_q$ *and let* $\mu = \sum_{j=0}^{m-1} p^j \mu_j$ $(\mu_j \in \Lambda_p)$ *be the* p*-adic expansion of* μ. *Then*

$$M(\mu) \simeq \bigotimes_{j=0}^{m-1} M(p^j\mu_j) \simeq \bigotimes_{j=0}^{m-1} \mathrm{Fr}^j(M(\mu_j)).$$

(If $m=\infty$, the factors in the tensor products are eventually K and so the products can be viewed as finite.)

The following proposition is well-known (cf. [7], p. 117).

1.2.2 PROPOSITION. *Assume* $m=\infty$ *so that* $G=G^{(\infty)}$. *Let* $\mu_1,\ldots,\mu_n \in \Lambda^+$ *and set* $\mu=\sum\mu_i$. *If* $M(\lambda)$ $(\lambda\in\Lambda^+)$ *is a composition factor of* $M = \otimes M(\mu_i)$, *then* $\lambda \leq \mu$. *Furthermore,* $M(\mu)$ *is a composition factor of* M *of multiplicity one.*

We use the tensor product theorem to strengthen this proposition when $m=\infty$ and to obtain for $m<\infty$ an analogous proposition which strengthens a result of Wong (see 2.6.2 and §2.7). This analog then leads to a recursion formula (3.1.1) for the Brauer characters afforded by the projective indecomposable modules for $G^{(m)}$ $(m<\infty)$; the formula generalizes one given earlier by Chastkofsky and Feit in their work on $SL(3,2^m)$. Finally, a "twisted" product formula (3.3.1) resembling 1.2.1 is obtained for a class of projective indecomposable characters and, by way of illustration, the character degrees are given for a few low rank groups (see §3.5).

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2. TENSOR PRODUCTS OF G-MODULES

Recall that m is fixed ($1 \leq m \leq \infty$) and $G=G^{(m)}$. Although most of what we do depends on m, we will usually make no explicit reference to it in the notation. Occasionally, we will need to work with both G and $G^{(\infty)}$ simultaneously; at those times, to avoid ambiguity, the notation associated with $G^{(\infty)}$ will bear the superscript (∞). Also, at times our arguments will need to be altered slightly to handle the case $m=\infty$. If a required adjustment is not obvious, it will be indicated.

2.1 The Grothendieck Algebra G

Let G be the Grothendieck algebra over the field $\mathbb{C}$ of complex numbers of the category of G-modules and let φ_M denote the element of G associated with the module M. If $m<\infty$, we view φ_M as the Brauer character afforded by M and thus identify G with the $\mathbb{C}$-algebra of class functions on the p-regular classes of G with values in $\mathbb{C}$. Instead of working exclusively with characters, we have introduced the Grothendieck algebra in order to simultaneously handle the case $m=\infty$.

For convenience, we will write $\varphi_{M(\lambda)}$ simply as φ_λ ($\lambda \in \Lambda_q$). As a $\mathbb{Z}$-module, G is free with basis $\{\varphi_\lambda | \lambda \in \Lambda_q\}$, the elements of which we will call <u>irreducible</u>. Each element φ of G can be written uniquely (up to order) as a $\mathbb{Z}$-linear combination of irreducibles: $\varphi = \sum_{\lambda \in \Lambda_q} a_\lambda \varphi_\lambda$ ($a_\lambda \in \mathbb{Z}$). We call $\sum a_\lambda$ the <u>length</u> of φ (written length(φ)) and a_λ the multiplicity of φ_λ in φ (written mult(φ_λ,φ)). If $\text{mult}(\varphi_\lambda,\varphi') \le \text{mult}(\varphi_\lambda,\varphi)$ for all $\lambda \in \Lambda_q$, we say that φ' is a constituent of φ and we write $\varphi' \subseteq \varphi$.

Note that if M is a G-module and if we write $\varphi_M = \sum a_\lambda\varphi_\lambda$, then each a_λ is nonnegative since a_λ is just the number of times that $M(\lambda)$ appears as a composition factor of M. Hence, in this case length(φ_M) is positive and equals the length of a composition series of M.

2.2 The Sets Λ^m and Λ^m_p.

Let $\Lambda^m = \bigoplus_{j=0}^{m-1} Y_j$ (weak direct sum if $m=\infty$), where Y_j is a copy of Λ. We view Y_j as a subgroup of Λ^m and denote by $\iota_j : \Lambda \to Y_j \subseteq \Lambda^m$ and $\pi_j : \Lambda^m \to Y_j \subseteq \Lambda^m$, the natural injection and projection, respectively. Also, when convenient we view Λ^m as a subset of Λ^∞ in the natural way.

Let $J = \{(i,j) \mid 1 \le i \le \ell,\ 0 \le j < m\}$, and for $(i,j) \in J$, set $\lambda_{ij} = \iota_j(\lambda_i)$. Then $\{\lambda_{ij} \mid (i,j) \in J\}$ is a $\mathbb{Z}$-basis for Λ^m and, with respect to this basis, Λ^m can be viewed as the set of $\ell\times m$-matrices over $\mathbb{Z}$ (eventually zero matrices if $m=\infty$).

Set $\alpha_{ij} = \iota_j(\alpha_i)$ and $\kappa_{ij} = p\lambda_{ij} - \lambda_{i(j+1)}$ (viewing second subscripts in $\mathbb{Z}/m\mathbb{Z}$ if $m<\infty$ so that $\lambda_{i(j+1)}$ is always defined). If we set $\mathcal{U} = \sum_{(i,j)\in J} \mathbb{Z}^+\alpha_{ij}$ and $\mathcal{H} = \sum_{(i,j)\in J} \mathbb{Z}^+\kappa_{ij}$, we obtain a partial order $\prec$ on Λ^m by taking $\mathcal{P} = \mathcal{U} + \mathcal{H}$ as the positive set and declaring $x' \prec x$ if $x - x' \in \mathcal{P}$. The relation is clearly reflexive and transitive so we need only prove antisymmetry. We require two lemmas; the proof of the first is outlined in [7], p. 72.

2.2.1 **LEMMA**. *Each* λ_i *is of the form* $\sum_j q_{ij}\alpha_j$, *where all* q_{ij} *are positive rational numbers.*

2.2.2 **LEMMA**. *If* $\sum a_{ij}\alpha_{ij} + \sum b_{ij}\kappa_{ij} = -\sum c_{ij}\lambda_{ij}$ *with* $a_{ij}, b_{ij}, c_{ij} \in \mathbb{Z}^+$, *then* $a_{ij} = b_{ij} = c_{ij} = 0$ *for all* $(i,j) \in J$.

Proof. Apply the homomorphism $(\lambda_{ij} \mapsto \lambda_i) : \Lambda^m \longrightarrow \Lambda$ to both sides of the equation to get $\sum_i(\sum_j a_{ij})\alpha_i = \sum_i(\sum_j (-c_{ij} - b_{ij}(p - 1)))\ \lambda_i$. By 2.2.1, the right hand side is a linear combination of the α_i's with nonpositive coefficients. Since $\{\alpha_i\}$ is linearly independent, the a_{ij}'s must all be zero. Now $\{\lambda_i\}$ is also linearly independent, so the b_{ij}'s and c_{ij}'s are also all zero. □

If $x \prec y$ and $y \prec x$ $(x,y \in \Lambda^m)$, then $y-x$ and $x-y$ are both in $\mathcal{P}$ and $(y-x) + (x-y) = 0$. 2.2.2 now implies that $x=y$ and thus $\prec$ is antisymmetric.

Let Λ_p^m denote $\sum_{j=0}^{m-1} \iota_j(\Lambda_p)$ (which can be viewed, relative to the basis $\{\lambda_{ij}\}$ of Λ^m, as the set of $\ell\times m$-matrices (a_{ij}) over $\mathbb{Z}^+$ with $0 \le a_{ij} \le p-1$). The map $wt : \lambda_{ij} \mapsto p^j\lambda_i$ defines a bijection of Λ_p^m onto Λ_q, the inverse being the map which sends $\sum a_i\lambda_i \in \Lambda_q$ to

$\sum a_{ij}\lambda_{ij}$, where $a_i = \sum a_{ij}p^j$ is the p-adic expansion of a_i. Therefore, since Λ_q indexes the irreducible modules, so does Λ_p^m. If $x \in \Lambda_p^m$, we write φ_x for $\varphi_{wt(x)}$ and note that $\{\varphi_x | x \in \Lambda_p^m\}$ is a $\mathbb{Z}$-basis for G.

The following theorem is practically a restatement of Steinberg's tensor product theorem (1.2.1) in the new notation. (We lose information in passing from modules to elements of the Grothendieck algebra, of course.)

2.2.3 THEOREM. *If* $x \in \Lambda_p^m$, *then* $\varphi_x = \prod_{j=0}^{m-1} \varphi_{\pi_j(x)}$.

Proof. Write $\pi_j(x) = \iota_j(\mu_j)$ with $\mu_j \in \Lambda_p$. Then $wt(x) = wt(\sum \iota_j(\mu_j)) = \sum p^j\mu_j$, so that $\varphi_x = \varphi_{wt(x)} = \prod \varphi_{p^j\mu_j} = \prod \varphi_{wt(\iota_j(\mu_j))} = \prod \varphi_{\iota_j(\mu_j)} = \prod \varphi_{\pi_j(x)}$, the second equality from 1.2.1. □

2.3 The Monoid $\mathfrak{X}$

Set $B_0 = \bigcup_{j=0}^{m-1} \iota_j(\Lambda_p)$ and let $\mathfrak{X}$ denote the free abelian monoid on the set $B = B_0\setminus\{0\}$. Thus, $\mathfrak{X}$ can be thought of as the multiplicative monoid consisting of formal products $x_1x_2\ldots x_n$ $(x_i \in B)$, with $x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(n)} = x_1x_2\ldots x_n$ for any permutation σ of $\{1,\ldots,n\}$ and with multiplication given by juxtaposition. We view B_0 as a subset of $\mathfrak{X}$ in the natural way (identifying $0 \in B_0$ with $1 \in \mathfrak{X}$).

When convenient, we also view $\mathfrak{X}$ as a subset of $\mathfrak{X}^{(\infty)}$ (see the first paragraph of this chapter for notation).

Let $x = x_1x_2\dots x_n \in \mathfrak{X}$ $(x_i \in B)$. Maps which were defined earlier give rise to induced homomorphisms on $\mathfrak{X}$ as follows:

$$\pi_j : x \mapsto \prod_i \pi_j(x_i) \in \mathfrak{X}, \quad \mathrm{wt} : x \mapsto \sum_i \mathrm{wt}(x_i) \in \Lambda^+ \quad \text{and} \quad \varphi_x = \prod \varphi_{x_i} \in G.$$

We also define a homomorphism $\mathfrak{X} \to (\Lambda^+)^m$ by $x \mapsto \bar{x} = \sum x_i$.

Let $\mathrm{len}(x) = n$ and set $\mathrm{ht}(x) = \max\{\mathrm{len}(\pi_j(x)) \mid 0 \le j < m\}$. The map $x \mapsto \bar{x}$ sets up a one-to-one correspondence between the set of elements in $\mathfrak{X}$ of height at most 1 and the set Λ_p^m; we identify these sets in the sequel. (Note that this identification causes no ambiguity with regard to the maps on $\mathfrak{X}$ and Λ_p^m which bear the same name. For instance, that φ_x is the same element of G whether we view x in Λ_p^m or in $\mathfrak{X}$ is the reformulation of Steinberg's tensor product theorem given in 2.2.3.)

Let $\mathrm{Fr}:G \to G$ be the homomorphism induced by $M \mapsto \mathrm{Fr}(M)$ (M, G-module) and let $\mathrm{Res}:G^{(\infty)} \to G$ be the homomorphism induced by $M \mapsto M|_G$ (M, $G^{(\infty)}$-module). We define homomorphisms on the set $\mathfrak{X}$ which are related to these and to do so we use the fact that each function f from B into a monoid Y induces a unique homomorphism from $\mathfrak{X}$ into Y which extends f. Let $\mathrm{fr} : \mathfrak{X} \to \mathfrak{X}$ be the homomorphism induced by $\iota_j(\mu) \mapsto \iota_{j+1}(\mu)$ (subscripts in $\mathbb{Z}/m\mathbb{Z}$ if $m < \infty$) and let $\mathrm{res} : \mathfrak{X}^{(\infty)} \to \mathfrak{X}$ be the homomorphism induced by $\iota_j(\mu) \mapsto \iota_{\bar{j}}(\mu)$ where $j \mapsto \bar{j}$ is the canonical map $\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ if $m < \infty$ and the identity $\mathbb{Z} \to \mathbb{Z}$ if $m = \infty$.

2.3.1 LEMMA.

(i) *For* $x \in \mathfrak{X}$ *we have* $\mathrm{Fr}(\varphi_x) = \varphi_{\mathrm{fr}(x)}$.

(ii) *For* $x \in \mathfrak{X}^{(\infty)}$ *we have* $\mathrm{Res}(\varphi_x^{(\infty)}) = \varphi_{\mathrm{res}(x)}$.

Proof. (i) Since Fr and fr preserve products, we may assume that $x \in B$, say $x = \iota_j(\mu)$ $(\mu \in \Lambda_p)$. If $j < m - 1$, then $\mathrm{Fr}(\varphi_{p^j\mu}) = \varphi_{p^{j+1}\mu} = \varphi_{\mathrm{wt}(\mathrm{fr}(x))}$, while, if $m < \infty$, then $\mathrm{Fr}(\varphi_{p^{m-1}\mu}) = \varphi_\mu = \varphi_{\mathrm{wt}(\mathrm{fr}(x))}$. Therefore, $\mathrm{Fr}(\varphi_x) = \mathrm{Fr}(\varphi_{\mathrm{wt}(x)}) = \mathrm{Fr}(\varphi_{p^j\mu}) = \varphi_{\mathrm{wt}(\mathrm{fr}(x))} = \varphi_{\mathrm{fr}(x)}$.

(ii) Similarly, since Res and res preserve products, we may assume that $x \in B$, say $x = \iota_j(\mu)$ $(\mu \in \Lambda_p)$. Write $x_0 = \iota_0(\mu)$ and observe that $x = \mathrm{fr}^j(x_0)$. By part (i) and the fact that $\mathrm{Res}\circ\mathrm{Fr}^j = \mathrm{Res}\circ\mathrm{Fr}^{\bar{j}}$, we obtain $\mathrm{Res}(\varphi_x^{(\infty)}) = \mathrm{Res}(\varphi^{(\infty)}_{\mathrm{fr}^j(x_0)}) = \mathrm{Res}(\varphi^{(\infty)}_{\mathrm{fr}^{\bar{j}}(x_0)}) = \varphi_{\mathrm{fr}^{\bar{j}}(x_0)} = \varphi_{\iota_{\bar{j}}(\mu)} = \varphi_{\mathrm{res}(x)}$. □

2.4 Decomposition of φ_x (Preliminaries)

In this section, we begin to address the problem of decomposing φ_x $(x \in \mathfrak{X})$ into a sum of elements of the form φ_y $(y \in \Lambda_p^m)$. As $\{\varphi_x | x \in \mathfrak{X}\}$ equals the submonoid of the multiplicative monoid $G\backslash\{0\}$ generated by $\{\varphi_\lambda | \lambda \in \Lambda_q\}$, what we are really investigating are the composition factors of a tensor product of irreducible modules.

For $0 \le j < m$, let $A_j = \{(a,b) \in \pi_j(\mathfrak{X}) \times \Lambda_p^\infty \mid \varphi_b^{(\infty)} \subseteq \varphi_a^{(\infty)}\}$. (An element of A_j corresponds to a choice of a composition factor in a tensor product of irreducible $G^{(\infty)}$-modules with restricted highest weights, each module twisted by Fr^j.)

The map fr acts on the elements of $\bigcup A_j$ componentwise. The following lemma is clear.

2.4.1 LEMMA. $A_j = fr^j(A_0)$ *for* $0 \le j < m$.

2.4.2 DEFINITIONS. For $\varsigma = (a,b) \in A_{j_0}$ $(0 \le j_0 < m)$, let

$$v(\varsigma) = \iota_{j_0}(p^{-j_0}(wt(a) - wt(b))) \in \mathcal{V},$$

$$h(\varsigma) = \sum_{i=1}^{\ell} \left(\sum_{j=j_0}^{\infty} \left(\sum_{k=j+1}^{\infty} b_{ik} p^{k-j-1} \right) \kappa_{ij} \right) \in \mathcal{H}, \quad \text{where } b = \sum b_{ij}\lambda_{ij}$$

(second subscript of κ_{ij} viewed in $\mathbb{Z}/m\mathbb{Z}$ if $m < \infty$), and

$$mult(\varsigma) = mult(\varphi_b^{(\infty)}, \varphi_a^{(\infty)}).$$

It is clear that $h(\varsigma)$ is in $\mathcal{H}$. We will show that $v(\varsigma)$ is in $\mathcal{V}$. By 2.4.1, $\varsigma = fr^{j_0}(\varsigma_0)$ for some $\varsigma_0 = (a_0, b_0) \in A_0$. Now, 1.2.2 implies $wt(a_0) - wt(b_0) =: \alpha \in \sum \mathbb{Z}^+\alpha_i$. So $wt(a) - wt(b) = p^{j_0}(wt(a_0) - wt(b_0)) = p^{j_0}\alpha$, whence $v(\varsigma) = \iota_{j_0}(\alpha) \in \mathcal{V}$.

2.5 Some Graphs

A <u>directed graph</u> Υ is a quadruple $(V^\Upsilon, E^\Upsilon, o, t)$ where $V = V^\Upsilon$ is a set of elements called <u>vertices</u>, $E = E^\Upsilon$ is a set of elements called <u>edges</u>, and o and t are maps from E into V. For $e \in E$, we call $o(e)$ the <u>original vertex</u> of e and $t(e)$ the <u>terminal vertex</u> of e. Let $v, v' \in V$. A <u>path</u> c (of length s) (from v to v') (with vertices v_i) is a sequence $e_1, \ldots, e_s$ of edges satisfying the following: $o(e_1) = v = v_0$, $t(e_s) = v' = v_s$, and $t(e_i) = o(e_{i+1}) = v_i$ $(1 \le i < s)$. The <u>essential length</u> (e.1.(c)) of the path c is the number of edges e_i for which $o(e_i) \ne t(e_i)$. The set of all paths of length s from v to v' is denoted by $C_s^\Upsilon(v,v')$ and we set $C^\Upsilon(v,v') = \bigcup_s C_s^\Upsilon(v,v')$. Finally, a relation L^Υ is defined on V by setting $v' L^\Upsilon v$ if $C^\Upsilon(v,v') \ne \phi$ (i.e. if there exists a path from v to v').

We define two particular directed graphs, Υ and Υ'.

(Υ) $V^\Upsilon = \mathfrak{X}$, $E^\Upsilon = \{(\zeta_0, \ldots, \zeta_{m-1}) \mid \zeta_j \in A_j\}$ and for $e = (\zeta_j) = ((a_j, b_j)) \in E^\Upsilon$, $o(e) = \prod a_j$ and $t(e) = \prod \operatorname{res}(b_j)$.

(Υ') $V^{\Upsilon'} = \mathfrak{X}$, $E^{\Upsilon'} = \{(\zeta, z) \mid \zeta \in \bigcup_{j=0}^{m-1} A_j,\ z \in \mathfrak{X}\}$ and for $e = (\zeta, z) = ((a,b),z) \in E^{\Upsilon'}$, $o(e) = az$ and $t(e) = \operatorname{res}(b)z$.

(If $m = \infty$, the elements of E^Υ are infinite sequences which are eventually $(1,1)$.)

Of the graphs Υ and Υ', Υ will play the more important role in what follows. The reason Υ' is introduced is that it is the easier of the two graphs to work with and, due to the following observation, certain statements about Υ' will carry over automatically to statements about Υ.

2.5.1 LEMMA. *If* $e = ((a_j, b_j))$ *is an edge in* Υ, *then*

$$((a_j, b_j), \prod_{k=j+1}^{m-1} a_k \prod_{k=0}^{j-1} \mathrm{res}(b_k))$$

$(0 \le j < m)$ *is a path in* Υ' *of length* m *from* $o(e)$ *to* $t(e)$ (*empty products being* 1). *In particular, if* $x' L^{\Upsilon} x$, *then* $x' L^{\Upsilon'} x$ $(x, x' \in \mathfrak{X})$.

Proof. The proof of the first statement is straightforward. For the second, we simply replace each edge of a path in Υ from x to x' with the path in Υ' described above. □

The path in Υ' constructed from a path c in Υ as in the proof above will be called the <u>path in</u> Υ' <u>associated</u> with c.

We extend the functions v, h and mult defined in 2.4.2: Let the values on $(\varsigma, z) \in E^{\Upsilon'}$ be those on ς; for a path $c' = e_1, \ldots, e_s$ in Υ' set $v(c') = \sum v(e_i)$, $h(c') = \sum h(e_i)$ and $\mathrm{mult}(c') = \prod \mathrm{mult}(e_i)$; and finally, define the values of these functions on a path c in Υ to be those on the path in Υ' associated with c.

Next, we investigate properties of the relations L^{Υ} and $L^{\Upsilon'}$.

2.5.2 LEMMA. *Let* $x, x' \in \mathfrak{X}$ *and assume* $x \in \Lambda_p^m$. *If* $x' L^{\Upsilon'} x$ (*resp.* $x' L^{\Upsilon} x$), *then* $x' = x$.

Proof. If $e = ((a,b),z)$ is an edge in Υ' with $az = o(e) = x$, then, since $ht(x) \leq 1$, we must have $a \in \Lambda_p^m$, whence $b = a$ and $t(e) = res(b)z = az = x$. The statement about Υ now follows from 2.5.1. □

We need a general technical lemma.

2.5.3 LEMMA. *Let* x_i, $i = 0,1,2,\ldots,$ *be elements of an arbitrary* $\mathbb{Z}$*-module and set* $y_i = px_i - x_{i+1}$. *If* $b \in \mathbb{Z}^+$ *has* p*-adic expansion* $b = \sum_{j=0}^{\infty} b_j p^j$ *and if we set* $c_0 = b - b_0$ *and* $c_j = -b_j$, $j > 0$, *then*

$$\sum_{j=0}^{\infty} c_j x_j = \sum_{j=0}^{\infty} \left(\sum_{k=j+1}^{\infty} b_k p^{k-j-1} \right) y_j .$$

Proof. We compare coefficients of x_j on both sides of the equation. If $j > 0$, this coefficient is c_j on the left and $p \sum_{k=j+1}^{\infty} b_k p^{k-j-1} - \sum_{k=j}^{\infty} b_k p^{k-j} = -b_j$ on the right. If $j = 0$, we get c_0 on the left and $p \sum_{k=1}^{\infty} b_k p^{k-1} = b - b_0$ on the right. □

Define $fr: \Lambda^m \to \Lambda^m$ by $\lambda_{ij} \mapsto \lambda_{i(j+1)}$ (second subscript in $\mathbb{Z}/m\mathbb{Z}$ if $m < \infty$) and $\overline{res}: \Lambda^\infty \to \Lambda^m$ by $\lambda_{ij} \mapsto \lambda_{i\bar{j}}$.

2.5.4 LEMMA. *If* c *is a path in* Υ' *(resp.* Υ*) from* x *to* x'

$(x, x' \in \mathfrak{X})$, *then* $\overline{x} - \overline{x}' = h(c) + v(c)$. *In particular,* $x' L^{\Upsilon'} x$ (*resp.* $x' L^{\Upsilon} x$) *implies* $\overline{x}' \prec \overline{x}$.

Proof. Because of the way h and v are defined on paths, we may assume c is of length 1, that is, c is an edge $((a,b),z)$ in Υ' from x to x'. Furthermore, since $x \mapsto \overline{x}$ is a homomorphism, we have $\overline{x} - \overline{x}' = (\overline{a} + \overline{z}) - (\overline{\mathrm{res}(b)} + \overline{z}) = \overline{a} - \overline{\mathrm{res}(b)}$, so we may as well assume $x = a$ and $x' = \mathrm{res}(b)$ (i.e. $z = 1$).

Now, $\varsigma = (a,b) \in A_{j_0}$ for some j_0; we first assume $j_0 = 0$. In this case, 2.4.2 reduces to $v(\varsigma) = \overline{a} - \iota_0(\mathrm{wt}(b))$. Upon writing $b = \sum b_{ij}\lambda_{ij}$ and $\mathrm{wt}(b) = \sum b_i\lambda_i$, we have $b_i = \sum_j p^j b_{ij}$ so that, by 2.5.3, we obtain

$$\overline{a} - b - v(\varsigma) = \iota_0(\mathrm{wt}(b)) - b = \sum_i b_i\lambda_{i0} - \sum_{i,j} b_{ij}\lambda_{ij}$$

$$= \sum_{i=1}^{\ell} \Big(\sum_{j=0}^{\infty} \Big(\sum_{k=j+1}^{\infty} b_{ik}p^{k-j-1} \Big) \kappa_{ij}^{(\infty)} \Big).$$

Applying $\overline{\mathrm{res}}$ to both sides and using the fact that $\overline{\mathrm{res}}(b) = \overline{\mathrm{res}}(\overline{b}) = \overline{\mathrm{res}(b)}$ we get $\overline{x} - \overline{x}' = \overline{a} - \overline{\mathrm{res}(b)} = h(\varsigma) + v(\varsigma) = h(c) + v(c)$.

It is a straightforward exercise to show that fr commutes with v, h, res and $x \mapsto \overline{x}$. Therefore, in view of 2.4.1, the general case follows from the special case.

Once again, the statement about Υ now follows from 2.5.1. □

2.5.5 LEMMA. *There exists a map* f *from* $\mathfrak{X}$ *into a well-ordered set* $(I,<)$ *having the property that if* $((a,b),z)$ *is an edge in* Υ' *from*

x *to* x', *then* $f(x') \leq f(x)$ *with strict inequality if* $\mathrm{len}(a)$ $(= \mathrm{ht}(a)) > 1$. *Thus, if* $x' L^{\Upsilon} x$, *then* $f(x') \leq f(x)$ *with strict inequality if* $\mathrm{ht}(x) > 1$.

Proof. Define $\mathrm{vol} : \Lambda^{m} \to \mathbb{Z}$ by $\sum a_{ij}\lambda_{ij} \mapsto \sum a_{ij}$ and let $\mathrm{vol}: \mathfrak{X} \to \mathbb{Z}^{+}$ be the induced homomorphism $\prod x_i \mapsto \sum \mathrm{vol}(x_i)$ $(x_i \in B)$.

For $x \in \mathfrak{X}$, set $f(x) = (\mathrm{length}(\varphi_x), \mathrm{len}(x) + \mathrm{vol}(x)) \in \mathbb{Z}^{+} \times \mathbb{Z}^{+} =: I$. I is well-ordered under the usual lexicographic ordering $<$: $(u,v) < (u',v')$ if $u < u'$ or $u = u'$ and $v < v'$.

To prove the first statement, note that since $\varphi_b^{(\infty)} \subseteq \varphi_a^{(\infty)}$, we have $\varphi_{x'} = \mathrm{Res}(\varphi_b^{(\infty)}\varphi_z^{(\infty)}) \subseteq \mathrm{Res}(\varphi_a^{(\infty)}\varphi_z^{(\infty)}) = \varphi_x$, so that $\mathrm{length}(\varphi_{x'}) \leq \mathrm{length}(\varphi_x)$.

Assume now that $\mathrm{length}(\varphi_{x'}) = \mathrm{length}(\varphi_x)$. Then, we must have $\varphi_b^{(\infty)} = \varphi_a^{(\infty)}$. If we write $a = \prod a_i$ with $a_i \in B$, then $\varphi_{\mathrm{wt}(b)}^{(\infty)} = \prod \varphi_{\mathrm{wt}(a_i)}^{(\infty)}$ so that the second statement of 1.2.2 implies $\mathrm{wt}(b) = \sum \mathrm{wt}(a_i) = \mathrm{wt}(a)$. Thus $v(\varsigma) = 0$ where $\varsigma = (a,b)$.

We need to show that $\mathrm{len}(x') + \mathrm{vol}(x') \leq \mathrm{len}(x) + \mathrm{vol}(x)$ with strict inequality in case $\mathrm{len}(a) > 1$. For this, we may assume $x = a$ and $x' = \mathrm{res}(b)$ (i.e., $z = 1$) since len and vol are homomorphisms. We also may assume $x \neq 1$, for $x = 1$ implies $x' = 1$ and both sides of the inequality become zero. In particular, we have $\mathrm{len}(x) \geq 1$.

Now, $\varsigma \in A_{j_0}$ for some $0 \leq j_0 < m$. Consider the case $j_0 = 0$. 2.5.4 gives

$$\mathrm{vol}(x) - \mathrm{vol}(x') = \mathrm{vol}(\bar{x} - \bar{x}') = \mathrm{vol}(h(\varsigma))$$

$$\geq \sum_{j=0}^{\infty} \left(\sum_{i=1}^{\ell} b_{i(j+1)}\right)(p - 1),$$

where $b = \sum b_{ij}\lambda_{ij}$. Since $\mathrm{len}(x') = \mathrm{len}(b) = |\{j | b_{ij} \neq 0 \text{ for some } i\}|$, we get $\mathrm{vol}(x) - \mathrm{vol}(x') \geq (\mathrm{len}(x') - 1)(p - 1) \geq \mathrm{len}(x') - 1$ whence $\mathrm{len}(x) + \mathrm{vol}(x) \geq \mathrm{len}(x') + \mathrm{vol}(x') - 1 + \mathrm{len}(x) \geq \mathrm{len}(x') + \mathrm{vol}(x')$ with the last inequality being strict if $\mathrm{len}(a) = \mathrm{len}(x) > 1$.

Since $\mathrm{len} \circ \mathrm{fr} = \mathrm{len}$ and $\mathrm{vol} \circ \mathrm{fr} = \mathrm{vol}$, the general case follows from the special case and 2.4.1.

For the second statement, first observe that we may assume there is an edge in Υ from x to x'. We can then use the first statement and 2.5.1 noting that $\mathrm{ht}(x) > 1$ implies $\mathrm{len}(a_j) > 1$ for some j, in the notation of that lemma. □

2.5.6 LEMMA. *Let* $x, x' \in \mathfrak{X}$ *and assume* $\bar{x} = \bar{x}' \in \Lambda_p^m$. *Then, there is no edge in* Υ *from* x *to* x' *unless* $x' = \bar{x}$, *in which case there is a unique such edge* e *and* $\mathrm{mult}(e) = 1$.

Proof. We first prove that if $e = ((a,b),z)$ is an edge of some path in Υ' from x to x', then $b = \bar{a}$. Let $x = x_0, \ldots, x_s = x'$ be the vertices of some path in Υ'. By 2.5.4, $\bar{x}_0 \succ \bar{x}_1 \succ \ldots \succ \bar{x}_s$. Since $\bar{x}_0 = \bar{x}_s$, antisymmetry of $\prec$ forces the equalities $\bar{x}_0 = \bar{x}_1 = \ldots = \bar{x}_s$. We may therefore assume that $s = 1$ and that e is an edge from x to x'. Now, $\bar{a} + \bar{z} = \bar{x} = \bar{x}' = \overline{\mathrm{res}(b)} + \bar{z}$, so that

$\overline{a} = \overline{\mathrm{res}(b)} = \overline{\mathrm{res}(\overline{b})}$. But 2.5.4 (with $m = \infty$) implies $\overline{a} - \overline{b} = \tau \in \mathfrak{p}^{(\infty)}$. Since $\overline{\mathrm{res}(\overline{a})} = \overline{a} = \overline{\mathrm{res}(\overline{b})}$ we have $\overline{\mathrm{res}(\tau)} = 0$. Thus, 2.2.2 implies that $\tau = 0$, whence $b = \overline{b} = \overline{a}$.

Returning to the proof of the lemma, we note that $e = ((\pi_j(x), \overline{\pi_j(x)}))$ $(0 \leq j < m)$ is an edge in Υ from $\prod \pi_j(x) = x$ to $\prod \mathrm{res}(\overline{\pi_j(x)}) = \prod \pi_j(\overline{x}) = \overline{x}$ and $\mathrm{mult}(e) = 1$. Indeed, since $\overline{x}$ is in Λ_p^m, so is $\pi_j(\overline{x}) = \overline{\pi_j(x)}$. Writing $\pi_j(x) = \prod x_i$ $(x_i \in B)$, we have $\mathrm{wt}(\overline{\pi_j(x)}) = \mathrm{wt}(\sum x_i) = \sum \mathrm{wt}(x_i)$. Thus, $\mathrm{mult}(\varphi^{(\infty)}_{\overline{\pi_j(x)}}, \varphi^{(\infty)}_{\pi_j(x)}) = \mathrm{mult}(\varphi^{(\infty)}_{\sum \mathrm{wt}(x_i)}, \prod \varphi^{(\infty)}_{\mathrm{wt}(x_i)}) = 1$ (1.2.2).

Conversely, any edge from x to x' must be of this form. To see this, let $e = ((a_j, b_j))$ be an arbitrary such edge. Then $x = \prod a_i$, so applying π_j we find that $a_j = \pi_j(x)$. By 2.5.1 and the first paragraph, we then have $b_j = \overline{a_j} = \overline{\pi_j(x)}$. □

2.5.7 COROLLARY. *Let* $x \in \mathfrak{X}$ *and assume* $\overline{x} \in \Lambda_p^m$. *Then, for each positive integer* s, *there is a unique path* c *in* Υ *of length* s *from* x *to* $\overline{x}$, *and* $\mathrm{mult}(c) = 1$.

Proof. If $x = x_0, x_1, \ldots, x_s = \overline{x}$ are the vertices of a path in Υ from x to $\overline{x}$, then $\overline{x}_0 = \overline{x}_1 = \ldots = \overline{x}_s$ (same proof as in 2.5.6). Using 2.5.6 repeatedly, we find that $\overline{x} = x_1 = x_2 = \ldots = x_s$, that for each i $(1 \leq i \leq s)$ there is a unique edge e_i in Υ from x_{i-1} to x_i, and that $\mathrm{mult}(e_i) = 1$. The corollary follows. □

2.6 Decomposition of φ_x

The next theorem is an expression for the multiplicity of an irreducible element of G as a constituent in a product of irreducibles (see opening paragraph of section 2.4).

For $x, x' \in \mathfrak{X}$, set $\text{e.l.}(x,x') = \text{lub}\{\text{e.l.}(c) \mid c \in C^{\Upsilon}(x,x')\}$.

2.6.1 THEOREM. *Let* $x \in \mathfrak{X}$ *and* $x' \in \Lambda_p^m$. *Then* $\text{e.l.}(x,x') < \infty$ *and for each positive integer* $s \geq \text{e.l.}(x,x')$ *we have*

$$\text{mult}(\varphi_{x'}, \varphi_x) = \sum_{c \in C_s^{\Upsilon}(x,x')} \text{mult}(c)$$

(an empty sum being interpreted as zero).

Proof. We proceed by (transfinite) induction on $f(x)$ where f is as in 2.5.5. First assume $\text{ht}(x) \leq 1$. Then, $x \in \Lambda_p^m$ and $\text{mult}(\varphi_{x'}, \varphi_x) = \delta_{x'x}$ (Kronecker delta). In this case $\text{e.l}(x,x')$ equals zero if $x' = x$ and equals $-\infty$ otherwise (2.5.2), so the theorem follows from 2.5.7.

Now assume $\text{ht}(x) > 1$. Then, since $x = \prod \pi_j(x)$, we obtain

$$\varphi_x^{(\infty)} = \prod_{j=0}^{m-1} \varphi_{\pi_j(x)}^{(\infty)} = \prod_{j=0}^{m-1} \Big(\sum_{y \in \Lambda_p^{\infty}} \text{mult}(\varphi_y^{(\infty)}, \varphi_{\pi_j(x)}^{(\infty)})\, \varphi_y^{(\infty)} \Big)$$

$$= \sum_{(y_j)} \Big(\prod_{j=0}^{m-1} \text{mult}(\varphi_{y_j}^{(\infty)}, \varphi_{\pi_j(x)}^{(\infty)}) \Big) \prod_{j=0}^{m-1} \varphi_{y_j}^{(\infty)}$$

where the sum is over all tuples $(y_0, \ldots, y_{m-1})$ with $y_j \in \Lambda_p^{\infty}$. Applying Res and using the definitions, we get

$$\varphi_x = \sum_{x''\in\mathfrak{X}} \sum_{c\in C_1^{\Upsilon}(x,x'')} \text{mult}(c)\, \varphi_{x''}.$$

Hence,

$$\text{mult}(\varphi_{x'},\varphi_x) = \sum_{x''\in\mathfrak{X}} \sum_{c\in C_1^{\Upsilon}(x,x'')} \text{mult}(c)\, \text{mult}(\varphi_{x'},\varphi_{x''}).$$

Note that if $C_1^{\Upsilon}(x,x'') \neq \phi$, then $f(x'') < f(x)$ by 2.5.5.

Since an element of $G^{(\infty)}$ can have only finitely many irreducible constituents, it follows that there are only finitely many edges in Υ with x as an original vertex, since each is of the form $((\pi_j(x),b_j))$ with $\varphi_{b_j}^{(\infty)} \subseteq \varphi_{\pi_j(x)}^{(\infty)}$ $(0 \leq j < m)$. This, together with the induction hypothesis, implies that $s' := \text{lub}\,\{\text{e.l.}(x'',x') \mid x'' \in \mathfrak{X},\ C_1^{\Upsilon}(x,x'') \neq \phi\} < \infty$. We also have that $\text{e.l.}(x,x') - 1 = s' \geq \text{e.l.}(x'',x')$ for each $x'' \in \mathfrak{X}$ with $C_1^{\Upsilon}(x,x'') \neq \phi$. (For the equality, we have used the fact that $x'' \neq x$ since, for instance, $f(x'') < f(x)$). Thus $\text{e.l.}(x,x') < \infty$ and, by the induction hypothesis,

$$\text{mult}(\varphi_{x'},\varphi_x) = \sum_{x''\in\mathfrak{X}} \sum_{c\in C_1^{\Upsilon}(x,x'')} \sum_{c'\in C_{s-1}^{\Upsilon}(x'',x')} \text{mult}(c)\, \text{mult}(c')$$

$$= \sum_{c\in C_1^{\Upsilon}(x,x')} \text{mult}(c),$$

as desired. □

2.6.2 COROLLARY. *If* $\varphi_{x'} \subseteq \varphi_x$ $(x' \in \Lambda_p^m,\ x \in \mathfrak{X})$, *then* $x' \prec \overline{x}$.

Proof. By 2.6.1, $x' L^{\Upsilon} x$, so 2.5.4 implies $x' = \overline{x}' \prec \overline{x}$. □

2.6.3 COROLLARY. *If* $x \in \mathfrak{X}$ *and* $\overline{x} \in \Lambda_p^m$, *then* $\mathrm{mult}(\varphi_{\overline{x}},\varphi_x) = 1$.

Proof. Use 2.6.1 and 2.5.7. □

We record another corollary for use in the next chapter. We need some notation. Let $\mathfrak{B} = \{\alpha_{ij},\kappa_{ij} | (i,j) \in J\}$ (= the set of generators of $\mathfrak{P}$) and for $\tau \in \mathfrak{P}$, set $\beta(\tau) = \{\beta \in \mathfrak{B} | \tau - \beta \in \mathfrak{P}\}$. Note that $\beta(\tau)$ is the set of all $\beta \in \mathfrak{B}$ which can possibly appear as a summand in an expression of τ as a sum of elements of $\mathfrak{B}$.

2.6.4 COROLLARY. *Assume* $m < \infty$. *Let* $x \in \mathfrak{X}$ *and* $x' \in \Lambda_p^m$ *with* $x' \prec \overline{x}$. *If* $\kappa_{i(m-1)} \notin \beta(\overline{x} - x')$ *for each* i $(1 \leq i \leq \ell)$, *then* $\mathrm{mult}(\varphi_{x'},\varphi_x) = \mathrm{mult}(\varphi_{x'}^{(\infty)},\varphi_x^{(\infty)})$.

Proof. The inequality $\mathrm{mult}(\varphi_{x'},\varphi_x) \geq \mathrm{mult}(\varphi_{x'}^{(\infty)},\varphi_x^{(\infty)})$ is clear, so it is enough to prove that $\mathrm{mult}(\varphi_{x'},\varphi_x) \leq \mathrm{mult}(\varphi_{x'}^{(\infty)},\varphi_x^{(\infty)})$. In view of 2.6.1 it suffices to show that every path in Υ from x to x' is also a path in $\Upsilon^{(\infty)}$ from x to x'.

Let $e = (\varsigma_j) = ((a_j,b_j))$ be an edge of a path in Υ from x to x'. Fix $0 \leq j_0 < m$. From 2.4.2 we have

$$h(\varsigma_{j_0}) = \sum_{i=1}^{\ell} \left(\sum_{j=j_0}^{\infty} \left(\sum_{k=j+1}^{\infty} b_{ik}p^{k-j-1} \right) \kappa_{ij} \right),$$

where $b_{j_0} = \sum b_{ij}\lambda_{ij}$. If b_{ij} were nonzero for some pair (i,j)

with $j \geq m$, then $\kappa_{i(m-1)}$ would appear with a nonzero coefficient in this sum. But this would contradict the assumption that $\kappa_{i(m-1)} \notin \beta(\overline{x} - x')$ in light of 2.5.4. Hence $b_{ij} = 0$ for all pairs (i,j) with $j \geq m$. Since j_0 was arbitrary, this shows that $b_j \in \Lambda_p^m$ for each j. Thus, $t(e) = \prod \text{res}(b_j) = \prod b_j = t^{(\infty)}(e)$. Since we always have $o(e) = o^{(\infty)}(e)$, we have shown that e is an edge in $\Upsilon^{(\infty)}$ from $o(e)$ to $t(e)$ and this completes the proof. □

2.7 Remarks

The second corollary (2.6.3) is nothing new; for $m = \infty$, it follows easily from 1.2.2, and for $m < \infty$, it was proved in Wong [12] (in the language of weights and using quite different methods, of course).

The first corollary (2.6.2) with $m = \infty$ generalizes the first statement in 1.2.2, and with $m < \infty$ generalizes a result of Wong. We will prove these assertions. In order to state Wong's result, we need to introduce some terminology. The set $\{\alpha_i, 1 \leq i \leq \ell\}$ of simple roots forms an $\mathbb{R}$-basis for the space $E = \sum \mathbb{R}\alpha_i$. A total ordering $<$ (called the lexicographic ordering) is imposed on E by declaring $\mu = \sum a_i\alpha_i$ $(a_i \in \mathbb{R})$ to be positive if the first nonzero a_i is positive.

2.7.1 PROPOSITION (Wong [12]). *Assume* $m < \infty$. *If* $\varphi_\lambda \subseteq \prod \varphi_{\mu_i}$ $(\lambda, \mu_i \in \Lambda_q)$, *then* $\lambda \leq \sum \mu_i$.

We need the following observations.

2.7.2 LEMMA.

(i) $wt(\mathcal{H}) = (q-1)\Lambda^+$ *if* $m < \infty$ *and* $wt(\mathcal{H}^{(\infty)}) = 0$, *and*

(ii) $wt(\mathcal{V}) = \sum \mathbb{Z}^+\alpha_i$ $(1 \le m \le \infty)$.

Proof. (i) If $m = \infty$, wt sends each κ_{ij} to 0; if $m < \infty$, wt sends κ_{ij} $(0 \le j < m-1)$ to 0 and $\kappa_{i(m-1)}$ to $(q-1)\lambda_i$. In either case, $\mathcal{H} = \sum \mathbb{Z}^+\kappa_{ij}$ so the result follows.

(ii) Obvious. □

Now start with the assumption $\varphi_\lambda \subseteq \prod \varphi_{\mu_i}$ $(\lambda, \mu_i \in \Lambda_q)$. Since wt maps Λ_p^m onto Λ_q we have $\lambda = wt(x')$ and $\mu_i = wt(x_i)$ for some $x, x_i \in \Lambda_p^m$. Setting $x = \prod x_i$, we have $\varphi_{x'} = \varphi_{wt(x')} = \varphi_\lambda \subseteq \prod \varphi_{wt(x_i)} = \prod \varphi_{x_i} = \varphi_x$. Therefore, 2.6.2 implies $x' \prec \bar{x}$, whence $\bar{x} - x' \in \mathcal{P} = \mathcal{H} + \mathcal{V}$. So, $\sum \mu_i - \lambda = wt(\bar{x}) - wt(x') \in wt(\mathcal{H}) + wt(\mathcal{V})$ and, in view of 2.7.2, this gives the first statement in 1.2.2 when $m = \infty$ and 2.7.1 when $m < \infty$. (The elements of $\sum \mathbb{Z}^+\alpha_i$ are obviously positive with respect to the lexicographic order and those of $(q-1)\Lambda^+$ are so by 2.2.1.)

3. PROJECTIVE INDECOMPOSABLE CHARACTERS

For this chapter, we assume $m < \infty$ and consider only the finite group $G = G^{(m)}$. (The standard results about Brauer characters can be found in Feit [5].)

3.1 A Recursion Formula

By the Krull-Schmidt theorem, the group ring KG, considered as a G-module via the regular representation, decomposes into a direct sum of projective indecomposable modules. Each projective indecomposable module has a unique irreducible quotient; this sets up a one-to-one correspondence between the isomorphism classes of projective indecomposable modules and the isomorphism classes of irreducible modules. For $x \in \Lambda_p^m$ we let Φ_x (or $\Phi_{wt(x)}$) denote the Brauer character afforded by the projective indecomposable module the irreducible quotient of which affords φ_x.

The Steinberg character is denoted Γ; it is the character afforded by the unique projective irreducible module. If we write $\gamma = \sum (p-1)\lambda_{ij}$, then $\Gamma = \Phi_\gamma = \varphi_\gamma$.

The symbol $(\, , \,)$ denotes the usual inner product of complex-valued functions defined on the set G_{reg} of p-regular elements of G: $(f,g) = \frac{1}{|G|} \sum_{s \in G_{reg}} f(s^{-1})g(s)$. It satisfies $(\Phi_x, \varphi_y) = \delta_{xy}$ (Kronecker delta) and $(fg,h) = (f,\bar{g}h)$, where bar signifies complex conjugation.

The following theorem generalizes a result of Chastkofsky and Feit in their work on $SL(3,2^m)$ (see [3], Lemma 7.1).

3.1.1 THEOREM. *Let* $x, w \in \Lambda_p^m$ *and* $z \in \mathfrak{X}$ *and assume* $\bar{z} = w - x$. *Then*

$$\Phi_x = \Phi_w \bar{\varphi}_z - \sum_{\substack{x < y \in \Lambda_p^m \\ y \neq x}} (\Phi_w, \varphi_z \varphi_y) \Phi_y .$$

Proof. Since Φ_w is the character afforded by a projective module, $\Phi_w \bar{\varphi}_z$ is as well. Thus, $\Phi_w \bar{\varphi}_z$ equals a sum of projective indecomposable characters. Consequently, we have

$$\Phi_w \bar{\varphi}_z = \sum_{y \in \Lambda_p^m} (\Phi_w \bar{\varphi}_z, \varphi_y) \Phi_y = \Phi_x + \sum_{\substack{x < y \in \Lambda_p^m \\ y \neq x}} (\Phi_w, \varphi_z \varphi_y) \Phi_y$$

since $0 \neq (\Phi_w, \varphi_z \varphi_y) = \mathrm{mult}(\varphi_w, \varphi_z \varphi_y)$ implies $y - x = y + \bar{z} - w \in \mathcal{P}$ (2.6.2) and $\mathrm{mult}(\varphi_w, \varphi_z \varphi_x) = 1$ (2.6.3). □

3.1.2 COROLLARY. *Let* $x \in \Lambda_p^m$. *Then*

$$\Phi_x = \Gamma \bar{\varphi}_{\gamma - x} - \sum_{\substack{x < y \in \Lambda_p^m \\ y \neq x}} (\Gamma, \varphi_{\gamma - x} \varphi_y) \Phi_y .$$

An immediate consequence of this corollary is the fact proved in Ballard [1] that the Steinberg character Γ divides every projective indecomposable character Φ_x. (This is obvious if x is maximal, since in this case the sum in 3.1.2 is empty. For the other x's we can use induction down the partial order of Λ_p^m.) Since the projective indecomposable characters comprise a basis for the vector space of complex-valued functions defined on the p-regular classes of G (see Feit [5], p. 146), it follows that Γ never vanishes. We will find it convenient to use the notation $\tilde{\Phi}_x = \Phi_x\Gamma^{-1}$.

3.2 A Factorization of $\tilde{\Phi}_x$

For a subset I of J, denote by π_I the homomorphism $\Lambda^m \to \Lambda^m$ which fixes λ_{ij} for $(i,j) \in I$ and which sends λ_{ij} to 0 for $(i,j) \notin I$, and set $\mathcal{B}(I) = \mathcal{B} \cap \pi_I(\Lambda^m)$. (Recall that $\mathcal{B} = \{\alpha_{ij},\kappa_{ij} | (i,j) \in J\}$.)

Let $\mathcal{J} = \{J_1,\ldots,J_s\}$ be a collection of subsets of J. For $x \in \Lambda^m_p$, define

$$\xi(x,\mathcal{J}) = \{y \in \Lambda^m_p | y \succ x \text{ and } \beta(y - x) \subseteq \bigcup \mathcal{B}(J_k)\}.$$

(Recall that $\beta(\tau) = \{\beta \in \mathcal{B} | \tau - \beta \in \mathcal{P}\}$.) If $\mathcal{J}$ contains a single set J_1, write $\xi(x,\mathcal{J})$ simply as $\xi(x,J_1)$ and set $\xi(x) = \xi(x,J)$ $(= \{y \in \Lambda^m_p | y \succ x\})$.

3.2.1 **LEMMA.** *Let* $\mathcal{J} = \{J_k\}$ *be a collection of subsets of* J, *let* $x \in \Lambda^m_p$ *and assume* $\xi(x) = \xi(x,\mathcal{J})$. *If* $x \prec y \in \Lambda^m_p$, *then* $\xi(y) = \xi(y,\mathcal{J})$.

Proof. Let $z \in \xi(y)$. By transitivity of $\prec$, we have $z \succ x$, whence $\beta(z - x) \subseteq \bigcup \mathcal{B}(J_k)$. But $\beta(z - x) = \beta(z - y + y - x) \supseteq \beta(z - y)$, so $z \in \xi(y,\mathcal{J})$. Thus, $\xi(y) \subseteq \xi(y,\mathcal{J})$. The other inclusion always holds. $\square$

A subset I of J is called a vertical subset if $(i,j) \in I$ implies $(i',j) \in I$ for each $1 \le i' \le \ell$. A vertical partition of J is a partition consisting of vertical subsets (i.e. a collection $\{J_k\}$ where J_k is a vertical subset of J, $\bigcup J_k = J$ and $J_k \cap J_{k'} = \phi$ for $k \neq k'$).

3.2.2 LEMMA. *Let* $\mathcal{J} = \{J_1, \ldots, J_s\}$ *be a vertical partition of* J *and let* $x \in \Lambda_p^m$. *For every* $y \in \xi(\gamma - x, \mathcal{J})$ *we have*

$$(\Gamma, \varphi_x \varphi_y) = \prod_k \mathrm{mult}(\varphi_{\gamma_k}, \varphi_{x_k} \varphi_{y_k})$$

where γ_k, x_k *and* y_k *are the images under* π_{J_k} *of* γ, x *and* y, *respectively.*

Proof. Because $\mathcal{J}$ is a vertical partition, it follows that $x = \prod x_k$ and $y = \prod y_k$. Therefore, since $x \mapsto \varphi_x$ is a homomorphism, we have

$$\varphi_x \varphi_y = \prod_k \varphi_{x_k} \varphi_{y_k} = \prod_k \Big(\sum_{z \in \Lambda_p^m} \mathrm{mult}(\varphi_z, \varphi_{x_k} \varphi_{y_k}) \, \varphi_z \Big).$$

Rearranging the product and sum, we obtain

$$\varphi_x \varphi_y = \sum_{(z_k)} \Big(\prod_k \mathrm{mult}(\varphi_{z_k}, \varphi_{x_k} \varphi_{y_k}) \, \varphi_{z_k} \Big),$$

where the sum is over all s-tuples $(z_1, \ldots, z_s)$ with $z_k \in \Lambda_p^m$. Therefore,

$$(\Gamma,\varphi_x\varphi_y) = \sum_{(z_k)} \left[\left(\prod_k \mathrm{mult}(\varphi_{z_k}, \varphi_{x_k}\varphi_{y_k}) \right) (\Gamma, \prod_k \varphi_{z_k}) \right].$$

Let (z_k) be a fixed tuple for which the corresponding term in the sum is nonzero. Then 2.6.2 implies

$$x_k + y_k - z_k =: \tau_k \in \mathcal{P} \quad \text{for each } k, \text{ and} \tag{3.2.3}$$

$$\sum z_k - \gamma =: \tau \in \mathcal{P}. \tag{3.2.4}$$

Adding equations 3.2.3 and 3.2.4 we obtain

$$x + y - \gamma = \sum \tau_k + \tau. \tag{3.2.5}$$

Now $\beta(x + y - \gamma) \subseteq \bigcup \mathcal{B}(J_i)$ by assumption, so that $\beta(\tau_k) \subseteq \bigcup \mathcal{B}(J_i)$ for each k, as well. But this implies that $\pi_{J_i}(\tau_k) \in \mathcal{P}$ for each i and k. Thus, if $i \neq k$, an application of π_{J_i} to 3.2.3 gives $-\pi_{J_i}(z_k) = \pi_{J_i}(\tau_k) \in -\Lambda_p^m \cap \mathcal{P} = \{0\}$ (2.2.2). Thus, τ_k and z_k are in $\pi_{J_k}(\Lambda^m)$ for each k. From 3.2.4, we then have $\gamma + \tau = \sum z_k \in \Lambda_p^m$, so that $\tau \in (\Lambda_p^m - \gamma) \cap \mathcal{P} = -\Lambda_p^m \cap \mathcal{P} = \{0\}$ (again by 2.2.2). Therefore, equation 3.2.4 becomes $\sum z_k = \gamma$ and, applying π_{J_k}, we find that $z_k = \gamma_k$ for each k. Finally, $(\Gamma, \prod \varphi_{z_k}) = (\Gamma,\varphi_\gamma) = 1$. □

3.2.6 COROLLARY. *Let* I *be a vertical subset of* J, *let* $x \in \Lambda_p^m$, *and assume that* $\pi_{J\setminus I}(x) = \pi_{J\setminus I}(\gamma)$. *If* $y \in \xi(x,I)$, *then*

$$(\Gamma,\varphi_{\gamma-x}\varphi_y) = \text{mult}(\varphi_{\gamma_0},\varphi_{\gamma_0-x_0}\varphi_{y_0})$$

where γ_0, x_0 *and* y_0 *are the images under* π_I *of* γ, x *and* y, *respectively.*

Proof. We apply the previous lemma with the partition $\{I,J\backslash I\}$ and note that since $\beta(y-x) \subseteq \mathcal{B}(I)$, we must have $\pi_{J\backslash I}(y-x) = 0$, whence $\pi_{J\backslash I}(y) = \pi_{J\backslash I}(x) = \pi_{J\backslash I}(\gamma)$. □

3.2.7 COROLLARY. *Let* I *be a vertical subset of* J *and let* $x \in \Lambda_p^m$. *Assume* $\pi_{J\backslash I}(x) = \pi_{J\backslash I}(\gamma)$ *and* $\xi(x) = \xi(x,I)$. *Then*

$$\tilde{\Phi}_x = \overline{\varphi}_{\gamma_0-x_0} - \sum_{\substack{x \prec y \in \Lambda_p^m \\ y \neq x}} \text{mult}(\varphi_{\gamma_0},\varphi_{\gamma_0-x_0}\varphi_{y_0})\,\tilde{\Phi}_y$$

where γ_0, x_0 *and* y_0 *are the images under* π_I *of* γ, x *and* y, *respectively.*

Proof. Use 3.2.6 and 3.1.2. □

3.2.8 DEFINITION. Let $\mathcal{J} = \{J_k\}$ be a vertical partition of J. Denote by $\Theta(\mathcal{J})$ the set of all $x \in \Lambda_p^m$ which satisfy

(i) $\xi(x) = \xi(x,\mathcal{J})$ and

(ii) $\xi(\tilde{x}_k) = \xi(\tilde{x}_k,J_k)$ for each k,

where $\tilde{x}_k = \pi_{J_k}(x) + \pi_{J\backslash J_k}(\gamma)$.

For the remainder of this section, we fix a vertical partition $\mathcal{J}$ and let the notation be as in 3.2.8.

3.2.9 LEMMA. *If* $x \in \Theta(\mathcal{J})$ *and* $x \prec y \in \Lambda_p^m$, *then* $y \in \Theta(\mathcal{J})$.

Proof. Condition (i) of 3.2.8 is handled by 3.2.1. Since $\xi(x) = \xi(x,\mathcal{J})$ we have $\beta(y - x) \subseteq \bigcup \mathcal{B}(J_k)$. It follows that $\tilde{y}_k - \tilde{x}_k = \pi_{J_k}(y - x) \in \mathcal{P}$ so that $\tilde{y}_k \succ \tilde{x}_k$ for each k (where $\tilde{y}_k = \pi_{J_k}(y) + \pi_{J\setminus J_k}(\gamma))$. Once again, 3.2.1 applies and condition (ii) of 3.2.8 is met. $\square$

3.2.10. LEMMA. *If* $x \in \Theta(\mathcal{J})$, *then the function* $f : \xi(x) \to \times_k \xi(\tilde{x}_k)$ *given by* $y \mapsto (\tilde{y}_k)$, *where* $\tilde{y}_k = \pi_{J_k}(y) + \pi_{J\setminus J_k}(\gamma)$, *is a bijection.*

Proof. The proof of 3.2.9 shows that $\tilde{y}_k \in \xi(\tilde{x}_k)$ so that f maps into $\times \xi(\tilde{x}_k)$. Define $g : \times \xi(\tilde{x}_k) \to \xi(x)$ by $(z_k) \mapsto z := \sum \pi_{J_k}(z_k)$. Since $z_k \in \xi(\tilde{x}_k) = \xi(\tilde{x}_k, J_k)$, we have $\beta(z_k - \tilde{x}_k) \subseteq \mathcal{B}(J_k)$, so that $z_k - \tilde{x}_k \in \mathcal{P} \cap \pi_{J_k}(\Lambda^m)$. Thus $z - x = \sum \pi_{J_k}(z_k - \tilde{x}_k) \in \mathcal{P}$, and $z \succ x$. Clearly f and g are inverses of each other. $\square$

3.2.11 THEOREM. *If* $x \in \Theta(\mathcal{J})$, *then* $\tilde{\Phi}_x = \prod \tilde{\Phi}_{\tilde{x}_k}$.

Proof. We proceed by induction down the partial order of Λ_p^m. If x is maximal, then $\xi(x) = \{x\}$. 3.2.10 then implies that $\xi(\tilde{x}_k) = \{\tilde{x}_k\}$ for each k. Applying 3.1.2, we have

$$\tilde{\Phi}_x = \overline{\varphi}_{\gamma-x} = \prod \overline{\varphi}_{\gamma_k - x_k} = \prod \tilde{\Phi}_{\tilde{x}_k} ,$$

where γ_k and x_k are the images under π_{J_k} of γ and x, respectively.

If x is not maximal, then 3.1.2 and 3.2.2 give

$$\tilde{\Phi}_x = \overline{\varphi}_{\gamma-x} - \sum_{\substack{x < y \in \Lambda_p^m \\ y \neq x}} \left(\prod_k \mathrm{mult}(\varphi_{\gamma_k}, \varphi_{\gamma_k - x_k} \varphi_{y_k}) \right) \tilde{\Phi}_y ,$$

where γ_k, x_k and y_k are the images under π_{J_k} of γ, x and y, respectively. By 3.2.9, each y in the sum lies in $\Theta(\mathcal{J})$, so the induction hypothesis and 3.2.10 imply

$$\tilde{\Phi}_x = \overline{\varphi}_{\gamma-x} - \sum_{(\tilde{y}_k)} \left(\prod_k \mathrm{mult}(\varphi_{\gamma_k}, \varphi_{\gamma_k - x_k} \varphi_{\pi_{J_k}(\tilde{y}_k)}) \, \tilde{\Phi}_{\tilde{y}_k} \right) + \prod_k \tilde{\Phi}_{\tilde{x}_k} ,$$

where the sum is over all $(\tilde{y}_k) \in \times \, \xi(\tilde{x}_k)$. (The last term occurs because we have included an extra term in the summation; the coefficient of this added term is one by 2.6.3.) Rearranging the sum and product, we obtain

$$\tilde{\Phi}_x = \overline{\varphi}_{\gamma-x} - \prod_k \left[\sum_{\tilde{x}_k < y \in \Lambda_p^m} \mathrm{mult}(\varphi_{\gamma_k}, \varphi_{\gamma_k - x_k} \varphi_{y_k}) \, \tilde{\Phi}_y \right] + \prod_k \tilde{\Phi}_{\tilde{x}_k} ,$$

where $y_k = \pi_{J_k}(y)$. Finally, using 3.2.7, we get

$$\tilde{\Phi}_x = \overline{\varphi}_{\gamma-x} - \prod_k \overline{\varphi}_{\gamma_k - x_k} + \prod_k \tilde{\Phi}_{\tilde{x}_k} = \prod_k \tilde{\Phi}_{\tilde{x}_k},$$

as desired. □

3.3 Twisted Products

Fix a tuple $(j_0,\ldots,j_s)$ $(1 \le s \le m)$ of integers with $0 = j_0 < j_1 < \ldots < j_{s-1} < j_s = m$. We define a vertical partition $\mathcal{J} = \{J_k\}$ of J associated with this tuple by setting $J_k = \{(i,j) \in J | j_k \le j < j_{k+1}\}$ $(0 \le k < s)$. If $j_k = k$ for each k, then $\mathcal{J}$ is called the column partition and is denoted $\mathcal{J}_c$.

Set $\delta_k = j_{k+1} - j_k$ and for $1 \le k \le m$, define $I_k = \{(i,j) \in J | 0 \le j < k\}$. Before stating the next result, which is a corollary of 3.2.11, we remark that $x \in \Lambda_p^m$ can be expressed (uniquely) in the form $\sum fr^{j_k}(y_k)$, where $y_k \in \pi_{I_{\delta_k}}(\Lambda_p^m)$. (Indeed, $y_k = fr^{m-j_k}(\pi_{J_k}(x))$.)

3.3.1. COROLLARY. *Assume* $x \in \Theta(\mathcal{J})$ *and write* x *in the form* $\sum fr^{j_k}(y_k)$ *with* $y_k \in \pi_{I_{\delta_k}}(\Lambda_p^m)$. *Then* $\tilde{\Phi}_x = \prod_{k=0}^{s-1} Fr^{j_k}(\tilde{\Phi}_{\tilde{y}_k})$, *where* $\tilde{y}_k = y_k + \pi_{J\setminus I_{\delta_k}}(\gamma)$. *Furthermore*, $\xi(\tilde{y}_k) = \xi(\tilde{y}_k, I_{\delta_k})$ *for each* k.

Proof. First note that if Q is a projective indecomposable module with irreducible quotient M, then $Fr(Q)$ is a projective indecomposable module with irreducible quotient $Fr(M)$. Thus, from 2.3.1 we get that $Fr(\tilde{\Phi}_z) = \tilde{\Phi}_{fr(z)}$ for any $z \in \Lambda_p^m$. Next, we observe that $\tilde{x}_k = \pi_{J_k}(x) + \pi_{J\backslash J_k}(\gamma) = fr^{j_k}(y_k + \pi_{J\backslash I_{\delta_k}}(\gamma)) = fr^{j_k}(\tilde{y}_k)$. The twisted product formula now follows from 3.2.11.

For the second statement, suppose $z \in \xi(\tilde{y}_k)$. If $z - \tilde{y}_k = \beta + \tau$ with $\beta \in \mathcal{B}\backslash\mathcal{B}(I_{\delta_k})$ and $\tau \in \mathcal{P}$, then applying fr^{j_k} we get $fr^{j_k}(z) - \tilde{x}_k = fr^{j_k}(\beta) + fr^{j_k}(\tau)$. Since $fr^{j_k}(\beta) \in \mathcal{B}\backslash\mathcal{B}(J_k)$ and $fr^{j_k}(\tau) \in fr^{j_k}(\mathcal{P}) = \mathcal{P}$, this contradicts that $\xi(\tilde{x}_k) = \xi(\tilde{x}_k, J_k)$. Hence, $\xi(\tilde{y}_k) \subseteq \xi(\tilde{y}_k, I_{\delta_k})$. Since the other inclusion always holds, we have equality. □

3.3.2 COROLLARY. *Assume* $m > 1$. *Let* $x \in \Theta(\mathcal{G}_c)$ *and write* $x = \sum \iota_j(\mu_j)$ $(\mu_j \in \Lambda_p)$. *Then* $\tilde{\Phi}_x = \prod_{j=0}^{m-1} Fr^j(\psi_{\mu_j})$, *where* ψ_μ $(\mu \in \Lambda_p)$ *is the (virtual) character given recursively (down the partial order in* Λ_p*) by*

$$\psi_\mu = \overline{\varphi}_{\sigma-\mu} - \sum_{\substack{\mu<\lambda\in\Lambda_p \\ \lambda\neq\mu}} \mathrm{mult}(\varphi_\sigma^{(\infty)}, \varphi_{\sigma-\mu}^{(\infty)}\varphi_\lambda^{(\infty)})\ \psi_\lambda,$$

where $\sigma = \sum (p-1)\lambda_i$. *In particular,* $\tilde{\Phi}_x(1) = \prod_{\mu\in\Lambda_p} \psi_\mu(1)^{d_\mu}$, *where* $d_\mu = |\{j \mid \mu_j = \mu\}|$.

Proof. By 3.3.1, $\tilde{\Phi}_x = \prod Fr^j(\tilde{\Phi}_{\tilde{y}_j})$, where $\tilde{y}_j = \iota_0(\mu_j) +$

$\pi_{J\setminus I_1}(\gamma)$. We will show that we can replace $\tilde{\Phi}_{\tilde{y}_j}$ with the indicated character.

Let $A = \{y(\mu) := \iota_0(\mu) + \pi_{J\setminus I_1}(\gamma) \mid \mu \in \Lambda_p$ and $\xi(y(\mu)) = \xi(y(\mu), I_1)\}$. We first show that $f : y = y(\mu) \mapsto \mu$ defines an injection $A \to \Lambda_p$ which sends $\xi(y)$ onto $\{\lambda \in \Lambda_p | \lambda \succ \mu\}$. The injection claim is clear. If $z \in \xi(y)$, then by 3.2.1, $\xi(z) = \xi(z, I_1)$. Also, $z - y$ is of the form $\sum a_i \alpha_{i0}$ $(a_i \in \mathbb{Z}^+)$ since the α_{i0}'s are the only elements of $\mathbb{B}$ which lie in $\pi_{I_1}(\Lambda^m)$. (Here we have used the assumption $m > 1$.) For one thing, this shows that $\pi_{J\setminus I_1}(z) = \pi_{J\setminus I_1}(y) = \pi_{J\setminus I_1}(\gamma)$, implying that $z \in A$, say $z = y(\lambda)$ $(\lambda \in \Lambda_p)$. We also get that $\lambda - \mu = wt(\iota_0(\lambda) - \iota_0(\mu)) = wt(z - y) = \sum a_i \alpha_i$, whence $\lambda \succ \mu$ and f maps $\xi(y)$ into $\{\lambda \in \Lambda_p | \lambda \succ \mu\}$. Finally, if $\mu \prec \eta \in \Lambda_p$, then $\eta - \mu = \sum b_i \alpha_i$ $(b_i \in \mathbb{Z}^+)$, so that $y(\eta) - y(\mu) = \iota_0(\sum b_i \alpha_i) = \sum b_i \alpha_{i0} \in \mathcal{P}$ and f maps $\xi(y)$ onto $\{\lambda \in \Lambda_p | \lambda \succ \mu\}$.

Now, since each $\tilde{y}_j$ is in A (see last statement in 3.3.1), it is enough to show that $y = y(\mu) \in A$ implies $\tilde{\Phi}_y = \psi_\mu$. For such a y, 3.2.7 implies

$$\tilde{\Phi}_y = \overline{\varphi}_{\gamma_0 - y_0} - \sum_{\substack{z \in \xi(y) \\ z \neq y}} \operatorname{mult}(\varphi^{(\infty)}_{\gamma_0}, \varphi^{(\infty)}_{\gamma_0 - y_0} \varphi^{(\infty)}_{z_0}) \, \tilde{\Phi}_z, \tag{3.3.3}$$

where γ_0, y_0 and z_0 are the images under π_{I_1} of γ, y and z, respectively. (Note that for $z \succ y$, we have $\beta(\gamma_0 - y_0 + z_0 - \gamma_0) =$

$\beta(z - y) \subseteq \mathcal{B}(I_1) = \{\alpha_{i0} | 1 \le i \le \ell\}$, so 2.6.4 allows us to insert the superscripts (∞).)

Assume y is maximal (in A). Then y is also maximal in Λ_p^m since, as we have shown above, $\xi(y) \subseteq A$. The second paragraph also implies that $\{\lambda \in \Lambda_p | \lambda \succ \mu, \lambda \ne \mu\} = \phi$. Thus, $\tilde{\Phi}_y = \overline{\varphi}_{\gamma_0 - y_0} = \overline{\varphi}_{wt(\gamma_0 - y_0)} = \overline{\varphi}_{\sigma-\mu} = \psi_\mu$.

Now if y is not maximal, we apply wt to all subscripts in 3.3.3 and use the second paragraph and induction down the partial order of A to get

$$\tilde{\Phi}_y = \overline{\varphi}_{\sigma-\mu} - \sum_{\substack{\mu \prec \lambda \in \Lambda_p \\ \lambda \ne \mu}} \mathrm{mult}(\varphi_\sigma^{(\infty)}, \varphi_{\sigma-\mu}^{(\infty)} \varphi_\lambda^{(\infty)}) \psi_\lambda,$$

which equals ψ_μ, as desired.

The formula for the degree $\tilde{\Phi}_x(1)$ follows immediately. $\square$

3.4 A Sufficient Condition for Membership in $\Theta(\mathcal{J}_c)$

Throughout this section, we will be dealing only with the column partition $\mathcal{J}_c = \{J_k\}$, where $J_k = \{(i,j) \in J | j = k\}$ $(0 \le k < m)$. If we let (c_{ij}) denote the Cartan matrix of Ψ and (d_{ij}) its inverse transpose, then $\alpha_i = \sum_j c_{ij} \lambda_j$ and $\lambda_i = \sum_j d_{ji} \alpha_j$. The next lemma is elementary.

3.4.1 LEMMA. *If* $\sum a_i \alpha_i = \sum b_i \lambda_i$ $(a_i, b_i \in \mathbb{R})$, *then* $a_i = \sum_j d_{ij} b_j$ *for each* i.

For the remainder of this section we fix an ℓ-tuple (c_i) of integers with $0 \le c_i < p$.

3.4.2 DEFINITION. Let $\mathfrak{A}$ denote the following hypothesis:

($\mathfrak{A}$1) Given ℓ-tuples (a_j) and (b_j) of nonnegative integers with $\sum a_j < \sum b_j$, there exists an i $(1 \le i \le \ell)$ such that $\sum_j d_{ij}(a_j - pb_j) + pd_{ii} \le 0$, and

($\mathfrak{A}$2) For any ℓ-tuples (a_j) and (b_j) of nonnegative integers with $\sum a_j = \sum b_j \ne 0$, there exists an i $(1 \le i \le \ell)$ such that $\sum_j d_{ij}(a_j - pb_j + c_j) < 0$.

These rather odd conditions are given with an eye toward the proof of 3.4.3. We will see in 3.4.10 that $\mathfrak{A}$ will be satisfied if, for instance, $(c_i) = (0)$ and p is roughly the rank of Ψ or larger.

Let $\mathfrak{Y}$ be the set of all $\sum y_i\lambda_i \in \Lambda_p$ such that $\sum_j d_{ij}y_j < d_{ii}p$ for each i $(1 \le i \le \ell)$.

3.4.3 THEOREM. *Assume* $\mathfrak{A}$ *is satisfied. Let* $x = \sum \iota_j(\mu_j)$ $(\mu_j \in \Lambda_p)$ *and set* $\sigma = \sum (p-1)\lambda_i$. *If* $\mu_j \in \sigma - \mathfrak{Y}$ *for each* j, *and* $\mu_{j_0} = \sum_i (p-1-c_i)\lambda_i$ *for some* j_0, *then* $x \in \Theta(\mathcal{G}_c)$.

Proof. Let $z \in \xi(x)$, and set $\tau = z - x \in \mathcal{P}$. We need to show first that $\beta(\tau) \subseteq \bigcup \mathcal{B}(J_k)$. We can write τ in two ways:

$$\sum a_{ij}\alpha_{ij} + \sum b_{ij}\kappa_{ij} = \tau = \sum t_{ij}\lambda_{ij} \tag{3.4.4}$$

where $a_{ij}, b_{ij} \in \mathbb{Z}^+$ and $t_{ij} \in \mathbb{Z}$ and the sums are over all $(i,j) \in J$. For any j $(0 \le j < m)$, we can apply π_j to 3.4.4 and get

$$\sum_i a_{ij}\alpha_{ij} = \sum_i (t_{ij} + b_{i,j-1} - pb_{ij})\,\lambda_{ij} \tag{3.4.5}$$

(second subscripts viewed in $\mathbb{Z}/m\mathbb{Z}$) so that by 3.4.1, we have

$$a_{ij} = \sum_k d_{ik}(t_{kj} + b_{k,j-1} - pb_{kj}) \tag{3.4.6}$$

for each $(i,j) \in J$.

If we write $x = \sum x_{ij}\lambda_{ij}$ $(x_{ij} \in \mathbb{Z}^+)$, then $\mu_j = \sum_i x_{ij}\lambda_i$, so that by assumption, $\sum_i y_{ij}\lambda_i \in \mathcal{Y}$, where $y_{ij} = p - 1 - x_{ij}$. Also, since $\tau = z - x \in \Lambda_p^m - x$, we have

$$t_{ij} \le p - 1 - x_{ij} = y_{ij} \tag{3.4.7}$$

for each $(i,j) \in J$.

We first show that $\sum_i b_{ij} = \sum_i b_{ij'}$ for $0 \le j,j' < m$. It is enough to show that $\sum_i b_{i,j-1} \ge \sum_i b_{ij}$ for each j (viewing second subscripts in $\mathbb{Z}/m\mathbb{Z}$). Suppose for some fixed j we have $\sum_i b_{i,j-1} < \sum_i b_{ij}$. Then by $\mathcal{Q}1$ of Definition 3.4.2, there exists an i for which (using 3.4.6, 3.4.7 and the definition of $\mathcal{Y}$)

$$a_{ij} \le \sum_k d_{ik}(y_{kj} + b_{k,j-1} - pb_{kj})$$

$$= (\sum_k d_{ik}y_{kj} - pd_{ii}) + (\sum_k d_{ik}(b_{k,j-1} - pb_{kj}) + pd_{ii}) < 0,$$

where we have used that $d_{ij} > 0$ for each (i,j) by 2.2.1. Since this contradicts that $a_{ij} \in \mathbb{Z}^+$, we conclude that $\sum_i b_{ij} = \sum_i b_{ij'}$ for $0 \le j,j' < m$.

We now prove that $\sum_i b_{ij_0} = 0$. Assume otherwise. From 3.4.7 we have $t_{ij_0} \le p - 1 - x_{ij_0} = c_i$, so that by 3.4.6 and $\mathfrak{A}2$ of Definition 3.4.2, there exists an i $(1 \le i \le \ell)$ such that

$$a_{ij_0} \le \sum_k d_{ik}(c_k + b_{k,j_0-1} - pb_{kj_0}) < 0,$$

which is again a contradiction. Thus, $\sum_i b_{ij_0} = 0$. By the previous paragraph, $\sum_i b_{ij} = 0$ for each j and, as $b_{ij} \ge 0$, we have $b_{ij} = 0$ for each $(i,j) \in J$. It follows that $\beta(\tau) \subseteq \bigcup \mathcal{B}(J_k)$, whence $z \in \xi(x,\mathcal{G}_c)$ and (i) of Definition 3.2.8 is met.

Next, we must show that $\xi(\tilde{x}_k) = \xi(\tilde{x}_k,J_k)$ for each k, where $\tilde{x}_k = \pi_{J_k}(x) + \pi_{J\setminus J_k}(\gamma)$. Let $z \in \xi(\tilde{x}_k)$ and set $\tau' = z - \tilde{x}_k \in \mathcal{P}$. As before, we can write τ' in two ways:

$$\sum a'_{ij}\alpha_{ij} + \sum b'_{ij}\kappa_{ij} = \tau' = \sum t'_{ij}\lambda_{ij} \qquad (3.4.8)$$

where $a'_{ij},b'_{ij} \in \mathbb{Z}^+$, $t'_{ij} \in \mathbb{Z}$ and the sums are over all $(i,j) \in J$. Now, as we have already observed, each d_{ij} is positive, so in particular, $\mathcal{Y}$ contains zero. Also, since $\mathfrak{A}2$ is satisfied, it is satisfied if we replace (c_i) with any other tuple (c'_i) where $0 \le c'_i \le c_i$ for each i (in particular, if each $c'_i = 0$). It follows that $\tilde{x}_k$ satisfies the hypothesis of the theorem. Therefore, by what we have shown above, each $b'_{ij} = 0$. Applying $\pi_{J\backslash J_k}$ to 3.4.8, we obtain

$$\sum_{j\neq k} a'_{ij}\alpha_{ij} = \sum_{j\neq k} t'_{ij}\lambda_{ij}.$$

For $j \neq k$, an argument similar to that leading to 3.4.7 gives $t'_{ij} \leq p - 1 - x'_{ij} = 0$ for each i, where $\tilde{x}_k = \sum x'_{ij}\lambda_{ij}$. Thus, by 2.2.2, $a'_{ij} = 0$ for all $j \neq k$. This proves that $\beta(\tau') \subseteq \mathbb{B}(J_k)$ so that $\xi(\tilde{x}_k) \subseteq \xi(\tilde{x}_k, J_k)$ as required in (ii) of Definition 3.2.8. □

For the remainder of the section we consider assumptions on Ψ, p and (c_i) which guarantee that $\mathbb{A}$ is satisfied.

3.4.9 LEMMA. *Let* $m_i = \min_j \{d_{ij}\}$ *and* $M_i = \max_j \{d_{ij}\}$.

(i) *If* $pm_i \geq M_i$ *for each* i, *then* $\mathbb{A}1$ *holds.*

(ii) *If* $pm_i > M_i$ *for some* i *and* $(c_i) = (0)$, *then* $\mathbb{A}2$ *holds.*

Proof. (i) In the notation of 3.4.2, choose any i with $b_i \neq 0$. Then $\sum_j d_{ij}(a_j - pb_j) + pd_{ii} = \sum_j d_{ij}a_j - p\sum_{j\neq i} d_{ij}b_j - pd_{ii}(b_i - 1) \leq M_i \sum a_j - pm_i \sum_{j\neq i} b_j - pm_i(b_i - 1) \leq M_i(\sum b_j - 1) - pm_i \sum b_j + pm_i = (M_i - pm_i)(\sum b_j - 1) \leq 0$.

(ii) For any i with $pm_i > M_i$, we have $\sum_j d_{ij}(a_j - pb_j + c_j) \leq M_i \sum a_j - pm_i \sum b_j = (M_i - pm_i) \sum b_j < 0$. □

3.4.10 COROLLARY. $\mathfrak{a}$ *is satisfied if* $(c_i) = (0)$ *and*

$$\begin{array}{llll} p \geq \ell+1 & \textit{if } \Psi = A_\ell, & p \geq 5 & \textit{if } \Psi = E_\ell, \\ p \geq \ell & \textit{if } \Psi = B_\ell \textit{ or } C_\ell, & p \geq 3 & \textit{if } \Psi = F_4, \\ p \geq \ell-1 & \textit{if } \Psi = D_\ell, & p \geq 2 & \textit{if } \Psi = G_2. \end{array}$$

To prove the corollary, one applies 3.4.9 to each root system. We refer the reader to the table in Humphreys [7], p. 69, in which the matrices $(c_{ij})^{-1} = {}^t(d_{ij})$ are given.

Some of the examples in the next section are not covered by 3.4.9. We provide separate arguments for them below.

3.4.11 LEMMA. *If* $\Psi = A_\ell$ *and* $\sum c_i < p-1$, *then* $\mathfrak{a}2$ *is satisfied.*

Proof. If not, then, since $d_{1j} = \frac{\ell+1-j}{\ell+1}$ and $d_{\ell j} = \frac{j}{\ell+1}$, we obtain for some ℓ and p the contradiction $0 \leq \sum_j d_{1j}(a_j + c_j - pb_j) + \sum_j d_{\ell j}(a_j + c_j - pb_j) = \sum (d_{1j} + d_{\ell j})(a_j + c_j - pb_j) = \sum a_j - \sum b_j - (p-1) \sum b_j + \sum c_j < 0$. □

3.4.12 LEMMA. *If* $(\Psi, p, (c_i)) = (A_3, 2, (0))$, *then* $\mathfrak{a}$ *is satisfied.*

Proof. $\mathfrak{a}2$ is handled by 3.4.11, so it suffices to show that $\mathfrak{a}1$ is satisfied; to do this, we consider various cases. (Note that

$$(d_{ij}) = \frac{1}{4} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

for type A_3.)

($b_1 \neq 0$ and $b_3 = 0$) In this case, $4(\sum_j d_{1j}(a_j - pb_j) + pd_{11}) = 3(a_1 - 2b_1) + 2(a_2 - 2b_2) + a_3 + 6 \leq 3 \sum a_j - 6b_1 - 4b_2 + 6 = 3 \sum a_j - 4b_1 - 2b_1 - 4b_2 + 6 \leq 3 \sum a_j - 4 \sum b_j + 4 \leq 3 \sum a_j - 4(\sum a_j + 1) + 4 = -\sum a_j \leq 0$, so that $\mathfrak{Q}1$ is satisfied with $i = 1$.

($b_1 = 0$ and $b_3 \neq 0$) The matrix (d_{ij}) is symmetric, so an argument similar to that in the previous case shows that $\mathfrak{Q}1$ is satisfied with $i = 3$.

($b_1 \neq 0$ and $b_3 \neq 0$) If $\sum_j d_{1j}(a_j - pb_j) + pd_{11} \leq 0$, then $\mathfrak{Q}1$ holds with $i = 1$, so assume otherwise. We have $\sum_j d_{3j}(a_j - pb_j) + pd_{33} < \sum_j d_{1j}(a_j - pb_j) + pd_{11} + \sum_j d_{3j}(a_j - pb_j) + pd_{33} = \sum (d_{1j} + d_{3j}) (a_j - pb_j) + pd_{11} + pd_{33} = \sum (a_j - pb_j) + pd_{11} + pd_{33} =$
$\sum a_j - 2 \sum b_j + 3 \leq (\sum b_j - 1) - 2 \sum b_j + 3 = -\sum b_j + 2 \leq 0$, so that $\mathfrak{Q}1$ is satisfied with $i = 3$.

($b_1 = 0$ and $b_3 = 0$) Here $\mathfrak{Q}1$ holds with $i = 2$ as $4(\sum d_{2j}(a_j - pb_j) + pd_{22}) = 2a_1 + 4(a_2 - 2b_2) + 2a_3 + 8 \leq 4 \sum a_j - 8 \sum b_j + 8 \leq 4 \sum a_j - 8(\sum a_j + 1) + 8 = -4 \sum a_j \leq 0$. $\square$

3.5 Examples

Here, we illustrate the results of the previous sections by computing the degrees of some projective indecomposable characters. The formulas in (i), (ii), (v) and (vi) below have already appeared in the literature and references are given.

3.5.1 PROPOSITION. *View* $x \in \Lambda_p^m$ *as a matrix (see section* 2.2) *and let* #[c] *denote the number of its columns which equal the column vector* [c]. *For the indicated groups, we have the following:*

(i) $SL(2,p^m)$, p *arbitrary,* $m > 1$: *If* $x \neq 0$, *then* $\Phi_x(1) = p^m 2^a$, *where* $a = m - \#[p-1]$ (*cf. Srinivasan* [10], *p.* 113).

(ii) $SL(3,2^m)$: *If* x *has at least one column which equals* $\begin{bmatrix}1\\1\end{bmatrix}$, *then* $\Phi_x(1) = 2^{3m}6^a3^b$, *where* $a = \#\begin{bmatrix}0\\0\end{bmatrix}$ *and* $b = \#\begin{bmatrix}1\\0\end{bmatrix} + \#\begin{bmatrix}0\\1\end{bmatrix}$ (*cf. Chastkofsky and Feit* [3], *p.* 136).

(iii) $SL(3,3^m)$: *If* x *has no zero column and at least one column which equals* $\begin{bmatrix}2\\2\end{bmatrix}$, *then* $\Phi_x(1) = 3^{3m}6^a3^b$, *where* $a = \#\begin{bmatrix}1\\0\end{bmatrix} + \#\begin{bmatrix}0\\1\end{bmatrix} + \#\begin{bmatrix}1\\1\end{bmatrix}$ *and* $b = \#\begin{bmatrix}2\\0\end{bmatrix} + \#\begin{bmatrix}0\\2\end{bmatrix} + \#\begin{bmatrix}2\\1\end{bmatrix} + \#\begin{bmatrix}1\\2\end{bmatrix}$.

(iv) $SL(4,2^m)$: *If* x *has no zero column and at least one column which equals* $\begin{bmatrix}1\\1\\1\end{bmatrix}$, *then* $\Phi_x(1) = 2^{6m}12^a6^b4^c$, *where* $a = \#\begin{bmatrix}1\\0\\0\end{bmatrix} + \#\begin{bmatrix}0\\0\\1\end{bmatrix} + \#\begin{bmatrix}0\\1\\0\end{bmatrix}$, $b = \#\begin{bmatrix}1\\0\\1\end{bmatrix}$ *and* $c = \#\begin{bmatrix}1\\1\\0\end{bmatrix} + \#\begin{bmatrix}0\\1\\1\end{bmatrix}$.

(v) $G_2(2^m)$: *If* x *has no zero column and at least one column which equals* $\begin{bmatrix}1\\1\end{bmatrix}$, *then* $\Phi_x(1) = 2^{6m}12^a6^b$, *where* $a = \#\begin{bmatrix}1\\0\end{bmatrix}$ *and* $b = \#\begin{bmatrix}0\\1\end{bmatrix}$ (*cf. Cheng* [4], *p.* 114).

(vi) $G_2(3^m)$: *If* x *has no column which equals* $\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\end{bmatrix}$, $\begin{bmatrix}1\\0\end{bmatrix}$ *or* $\begin{bmatrix}2\\0\end{bmatrix}$ *and at least one column which equals* $\begin{bmatrix}2\\2\end{bmatrix}$, *then* $\Phi_x(1) = 3^{6m}36^a12^b6^c$, *where* $a = \#\begin{bmatrix}1\\1\end{bmatrix}$, $b = \#\begin{bmatrix}0\\2\end{bmatrix}$ *and* $c = \#\begin{bmatrix}2\\1\end{bmatrix} + \#\begin{bmatrix}1\\2\end{bmatrix}$ (*cf. Cheng* [4], *p.* 84).

In each case, the restrictions on x are those appearing in 3.4.3; they guarantee that $x \in \Theta(\mathcal{G}_c)$. We sketch the derivation of the formula in (iv) and remark that the other derivations are similar. (For further details, see [6].)

Assume $G = SL(4,2^m)$. In the following discussion, we identify the set Λ_p with the set of integers i, $0 \leq i < 8$ via $a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 \leftrightarrow a_1 + 2a_2 + 4a_3$.

3.5.2 LEMMA.

(i) $\overline{\varphi}_1 = \varphi_4$, $\overline{\varphi}_2 = \varphi_2$, $\overline{\varphi}_5 = \varphi_5$, $\overline{\varphi}_3 = \varphi_6$ *and* $\overline{\varphi}_7 = \varphi_7$.

(ii) $\varphi_1(1) = \varphi_4(1) = 4$, $\varphi_2(1) = 6$, $\varphi_5(1) = 14$, $\varphi_3(1) = \varphi_6(1) = 20$ *and* $\varphi_7(1) = 64$.

(iii) $\mathrm{mult}(\varphi_7^{(\infty)}, \varphi_5^{(\infty)}\varphi_7^{(\infty)}) = \mathrm{mult}(\varphi_7^{(\infty)}, \varphi_3^{(\infty)}\varphi_3^{(\infty)}) = \mathrm{mult}(\varphi_7^{(\infty)}, \varphi_6^{(\infty)}\varphi_6^{(\infty)}) = 2$.

The module $M(\lambda)$ $(\lambda\in\Lambda^+)$ for the infinite group $G^{(\infty)} = SL(4,K)$ satisfies $M(\lambda)^* \simeq M(-\omega_0\lambda)$ where $M(\lambda)^*$ denotes the contragredient of $M(\lambda)$ and ω_0 denotes the longest element of the Weyl group of Ψ (see Steinberg [11], p. 213). In the present situation, $-\omega_0$ exchanges λ_1 with λ_3 and fixes λ_2. This gives (i).

Let $\lambda \in \Lambda^+$ and let $V(\lambda)_{\mathbb{C}}$ denote an irreducible module of highest weight λ for the simple Lie algebra over $\mathbb{C}$ of type Ψ. By tensoring a minimal admissible lattice in $V(\lambda)_{\mathbb{C}}$ with K we obtain a $G^{(\infty)}$-module $V(\lambda)$ (called a Weyl module). $V(\lambda)$ possesses a "contravariant form" the kernel N of which is the unique maximal submodule of $V(\lambda)$. Furthermore, $V(\lambda)/N \simeq M(\lambda)$ (see Wong [11], p. 362). By using Freudenthal's formula or by writing down semistandard Young tableaux as in James and Kerber [8], one can compute the formal character $ch(\lambda)$ of $V(\lambda)$. Then, by inspecting the (Gram) matrix of the contravariant form, one can determine which weights (with multiplicity) are lost in passing to the quotient $V(\lambda)/N$ and thus determine the formal character $p\text{-}ch(\lambda)$ of $M(\lambda)$. In particular, this process gives the degree formulas in (ii). (Although the method described is not practical in general, it is suitable for our present needs.)

Finally, we turn to the multiplicity formulas in (iii). Given $\lambda, \lambda' \in \Lambda^+$, the formal character of $M(\lambda) \otimes M(\lambda')$ is $(p\text{-}ch(\lambda))(p\text{-}ch(\lambda'))$. Also, the set $\{p\text{-}ch(\mu) | \mu \in \Lambda^+\}$ is a $\mathbb{Z}$-basis for the set of elements in the group ring $\mathbb{Z}[\Lambda]$ which are invariant under the Weyl group (see Bourbaki [2], chap. VI, §3, no. 4). It follows that if one can find a decomposition $(p\text{-}ch(\lambda))(p\text{-}ch(\lambda')) = \sum p\text{-}ch(\mu_i)$ $(\mu_i \in \Lambda^+)$, then the $M(\mu_i)$ must be the composition factors of $M(\lambda) \otimes M(\lambda')$. In [6], the composition factors of the products $M(\lambda) \otimes M(\lambda')$ with $\lambda, \lambda' \in \Lambda_p$ were computed by applying the described method to a few of the products to get started (using parts (i) and (ii) and 2.6.2 to keep trial and error to a minimum) and then by applying associativity of tensor products (writing appropriate three-fold tensor products in two ways) to obtain the composition factors of the remaining products.

We now return to the proof of 3.5.1 (iv). If $m = 1$, the formula is obvious, so assume $m > 1$. The partial order lattice in Λ_p is $0 \prec 5$, $1 \prec 6$, $4 \prec 3$ and $2 \prec 7$. Using 3.5.2 we find that $\psi_7(1) = 1$, $\psi_6(1) = 4$, $\psi_5(1) = 6$, $\psi_4(1) = 12$, $\psi_3(1) = 4$, $\psi_2(1) = 12$ and $\psi_1(1) = 12$ in the notation of 3.3.2. Now $\mathcal{Y}$ is the set of all $y_1\lambda_1 + y_2\lambda_2 + y_3\lambda_3 \in \Lambda_p$ satisfying $3y_1 + 2y_2 + y_3 < 6$, $2y_1 + 4y_2 + 2y_3 < 8$ and $y_1 + 2y_2 + 3y_3 < 6$, so that $\mathcal{Y} = \Lambda_p \backslash \{7\}$. By 3.4.12 $\mathfrak{Q}$ is satisfied if $(c_i) = (0)$, so 3.4.3 implies $x \in \Theta(\mathcal{G}_c)$. The degree formula now follows from 3.3.2 after we note that the degree of the Steinberg character is 2^{6m} (by 3.5.2 (ii) and 1.2.1).

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